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Finite-Dimensional Quantum Mechanics of a Particle. II

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The finite-dimensional quantum mechanics (FDQM) based on Weyl's form of Heisenberg's canonical commutation relations, developed for the case of one-dimensional space, is extended to three-dimensional space. This FDQM is applicable to the physics of particles confined to move within finite regions of space and is significantly different from the current quantum mechanics in the case of atomic and subatomic particles only when the region of confinement is extremely small—of the order of nuclear or even smaller dimensions. The configuration space of such a particle has a quantized eigenstructure with a characteristic dependence on its rest mass and dimension of the region of confinement, and the current Schrödinger–Heisenberg formalism of quantum mechanics becomes an asymptotic approximation of this FDQM. As an example a spherical harmonic oscillator with a particular radius of confinement is considered.

1. INTRODUCTION

In a recent contribution to this journal by us (Jagannathan, Santhanam and Vasudevan, 1981 hereafter referred to as I) the form of quantum mechanics in which the Schrödinger operators for position and momentum obeying Heisenberg's canonical commutation relations are replaced by finite-dimensional matrices consistent with Weyl's form of Heisenberg's relation has been called finite-dimensional quantum mechanics (FDQM) and the one-dimensional version of such a FDQM has been developed. This paper continues the discussion of the mathematical structure and the physical significance of the FDQM with the generalization of the theory for the case of the three-dimensional space. Essentially, in our opinion, FDQM, instead of the usual Schrödinger–Heisenberg formalism of quantum mechanics, would give the correct description of the physics of a particle

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totally confined to move within a finite region of space whatever may be the forces causing such a confinement, and hence this study must be of significance in the present stage of development of the theory of quarks. Then the current quantum mechanical formalism, in which for such a case coordinate and momentum operators with unbounded spectra are used along with suitable boundary conditions, would be an asymptotic approximation.

Though Weyl (1932) had expressed the view that the finite-dimensional representations of his form of Heisenberg's canonical commutation relations might play a fundamental role in the future development of quantum mechanics, particularly in the realm of nuclear physics, his work and the subsequent work of Schwinger (1960) employed such finite-dimensional representations of the unitary operators $\hat{U}_\alpha$ and $\hat{V}_\beta$ defined by

$$\hat{U}_\alpha = \exp(i\alpha\hat{p}/\hbar), \qquad \hat{p}f(q) = -i\hbar\frac{\partial f}{\partial q} \tag{1}$$

$$\hat{V}_\beta = \exp(i\beta\hat{q}/\hbar), \qquad \hat{q}f(q) = qf(q) \tag{2}$$

for a suitable set of values of the parameters α and β only in the intermediate step towards an understanding of the algebraic structure of the current quantum kinematics. Many direct physical applications of the finite-dimensional forms of the operators obeying the Weyl commutation relation

$$\hat{U}_\alpha\hat{V}_\beta = \exp(i\alpha\beta/\hbar)\hat{V}_\beta\hat{U}_\alpha \tag{3}$$

have been considered by Alladi Ramakrishnan and his collaborators (Alladi Ramakrishnan, 1972) as part of an extensive analysis of generalized Clifford algebras (see for instance, Alladi Ramakrishnan, 1971, 1972 for the mathematical literature on generalized Clifford algebras). Only very recently the possible fundamental implications of interpreting the finite-dimensional representations of the operators $\hat{q}$ and $\hat{p}$ consistent with (1)–(3), respectively, as the actual position and momentum operators of a particle have been analyzed fully and a FDQM has been developed for one-dimensional space by us in I starting from certain essential modifications of the ideas on the form of a quantum mechanics in discrete space discussed by one of us in detail (see Santhanam and Tekumalla, 1976; Santhanam 1977a, 1977b, 1978). Barut and Bracken (1980) have also considered a particular generalization of the earlier work of one of us (Santhanam, 1977a) on the form of quantum mechanics in discrete one-dimensional space to the case of three-dimensional space. Our approach to the construction of the FDQM in three-dimensional space, presented here, is based on I and differs completely from the work of Barut and Bracken (1980).

Discretization of space-time has been discussed by several physicists for a long time from different points of view and in particular with reference to particle physics (see, for instance, Finkelstein, 1974; Lorente, 1974; Ginsburg, 1976; Dadic and Pisk, 1979; Stovicek and Tolar, 1979; Tati, 1980; Saavedra and Utreras 1981; for recent discussions of the subject and detailed references to earlier literature). As already emphasized in I the basic difference between our ideas and the earlier ideas is that in our work the configuration space of any particular particle is considered to have an eigenstructure characteristically dependent on its mass and the possible extent of its motion in space and time is considered to be an independent continuous parameter as it is in the current quantum theory, instead of an absolutely quantized space-time common to all material particles with universal minima of length and time or some kind of lattice structure. Also independent coordinate operators are assumed to commute as usual in the current quantum theory (see Snyder, 1947; Barut and Bracken, 1980, for instances of assumptions of different kinds).

In Section 2 we shall briefly review the one-dimensional formulation of the FDQM presented in I; in Section 3 we shall consider the basic mathematical structure of FDQM in three-dimensional space, and in section 4 we shall conclude with the example of the derivation of the energy spectrum of a spherical oscillator enclosed in a spherical region of a particular radius ρ according to our new theory.

2. FDQM IN ONE-DIMENSIONAL SPACE

The formalism of FDQM as established in I can be briefly summarized as follows. If a particle of rest mass m is confined to move within a one-dimensional region of finite length L then Weyl's form (1)–(3) of the Heisenberg commutation relation,

$$[\hat{q}, \hat{p}] = i\hbar \tag{4}$$

may be interpreted to imply that the dynamics of the particle depends on a FDQM based on the following set of basic principles:

(i) The position eigenvalues of the particle form a discrete and finite set $\{q_{Jn}\}$ given by

$$q_{Jn} = n\epsilon_J; \qquad n = -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \tag{5}$$

$$\epsilon_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} \lambda\!\!\!^{-}; \qquad \lambda\!\!\!^{-} = \frac{\lambda_c}{2\pi} = \frac{\hbar}{mc} \tag{6}$$

and characterized by an integer $J \geq 1$, the "space quantum number," such that

$$2J\epsilon_J \leq L < 2(J+1)\epsilon_{J+1} \tag{7}$$

or

$$J = \text{integer part of } \left[x + (x + x^2)^{1/2}\right] \qquad \text{with } x = \left(\frac{L^2}{8\pi\lambda^2}\right) \tag{8}$$

when there is no obvious distinction between positive and negative directions with respect to the center of the region of length L taken as the origin of the coordinate system in the frame of reference in which the concerned region is at rest. In other words the quantum mechanical system space of the particle is a $(2J+1)$-dimensional vector space characterized by a unique integer-valued space quantum number J given by (8). Corresponding particle operators are $(2J+1)$-dimensional matrices and the basic position operator Q_J, for the given value of J, is defined by

$$\begin{aligned} &\langle n|Q_J|n'\rangle = n\epsilon_J \delta_{nn'} \\ &n, n' = -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \end{aligned} \tag{9}$$

in the position representation where ϵ_J is given by (6) and $\langle n|\cdot|n'\rangle$ denotes the nn'-th matrix element as usual in the Dirac notation.

(ii) The momentum operator P_J conjugate to Q_J is given in position representation by

$$\begin{aligned} \langle n|P_J|n'\rangle &= \frac{\eta_J}{2J+1} \sum_{s=-J}^{J} s \exp\left[\frac{i2\pi s(n-n')}{2J+1}\right] \\ &= \begin{cases} 0 & \text{if } n = n', \\ \dfrac{i\eta_J}{2}\operatorname{cosec}\left[\dfrac{2\pi J(n-n')}{2J+1}\right] & \text{if } n \neq n' \end{cases} \\ n, n' &= -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \end{aligned} \tag{10}$$

$$\eta_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} mc \tag{11}$$

and consequently the momentum eigenvalues of the particle also form a discrete and finite set $\{p_{Jn}\}$ given by

$$p_{Jn} = n\eta_J, \qquad n = -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \tag{12}$$

(iii) If an observable K of the particle is represented by the operator $\hat{K}(\hat{q}, \hat{p})$ in the normal quantum mechanical formalism then it will now be represented by the matrix operator $K_J(Q_J, P_J)$ obtained from $\hat{K}(\hat{q}, \hat{p})$ by the rule of replacement

$$\hat{q} \to Q_J, \qquad \hat{p} \to P_J \tag{13}$$

with J chosen as in (8) and the eigenvalues of the matrix K_J are the values that the observable K can take. For example

$$\mathcal{H}_J = \frac{P_J^2}{2m} + \frac{1}{2} m\omega^2 Q_J^2$$

will represent the Hamiltonian operator corresponding to the nonrelativistic linear harmonic oscillation of the particle with space quantum number J and frequency ω and the eigenvalues and eigenvectors of the matrix $\mathcal{H}_J$ will characterize the corresponding energy eigenstates.

(iv) Except for the replacement of the usual infinite-dimensional Schrödinger–Heisenberg operators by finite-dimensional matrices and the integrations over space by suitable finite summations all other aspects of the current quantum theory are valid in general. For example, Born's probabilistic interpretation of the state vectors is applicable, expectation values of the observables can be defined in the usual manner, Heisenberg's uncertainty principle exists since Q_J and P_J do not commute, time is regarded as an independent continuous parameter, and the temporal development of the $(2J+1)$-component state vector $|\Psi_J(t)\rangle$ is governed by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\Psi_J(t)\rangle = \mathcal{H}_J |\Psi_J(t)\rangle \tag{14}$$

with $\mathcal{H}_J$ as the Hamiltonian matrix or for any observable K the rate of change of the corresponding matrix operator in the Heisenberg picture can be defined by

$$\left(\frac{dK}{dt}\right)_J = \left(\frac{\partial K_J}{\partial t}\right) + \frac{i}{\hbar}[\mathcal{H}_J, K_J] \tag{15}$$

Let us conclude this section with the following observations:

(a) The two simplest dimensionless constants $(\epsilon_J mc/\hbar)$ and (η_J/mc) that can be derived from the fundamental quantities, quantum of position ϵ_J, quantum of momentum η_J, rest mass of the particle m, Planck's constant

$\hbar$, and the speed of light c are of equal magnitude given by

$$\frac{\epsilon_J mc}{\hbar} = \frac{\eta_J}{mc} = \left(\frac{2\pi}{2J+1}\right)^{1/2} \tag{16}$$

as seen from (6) and (11) so that

$$\frac{\epsilon_J \eta_J}{\hbar} = \frac{2\pi}{2J+1} \tag{17}$$

and consequently Q_J and P_J given by (9) and (10) are canonically conjugate in the sense that

$$\begin{gathered}\exp(i\alpha P_J/\hbar)\exp(i\beta Q_J/\hbar) = \exp(i\alpha\beta/\hbar)\exp(i\beta Q_J/\hbar)\exp(i\alpha P_J/\hbar) \\ \alpha = n\epsilon_J, \qquad \beta = n'\eta_J \\ n, n' = 0, \pm 1, \ldots, \pm(2J-1), \pm 2J \end{gathered} \tag{18}$$

i.e., if we make the replacement, $\hat{q} \to Q_J$ and $\hat{p} \to P_J$, then Weyl's relation given by (1)–(3) is satisfied whenever α and β, respectively, represent the shifts in position and momentum eigenvalues.

(b) It follows from (6), (8), and (11) that

$$\text{as } L \to \infty, \qquad J \to \infty, \qquad \epsilon_J \to 0, \qquad \eta_J \to 0 \tag{19}$$

Now it can be verified easily that in this limit the spectra of position and momentum eigenvalues given in (5) and (12), respectively, become continuous ranging from $-\infty$ to ∞ and also

$$\lim_{J \to \infty} Q_J|\Psi_J\rangle \to \hat{q}|\Psi\rangle \tag{20}$$

$$\lim_{J \to \infty} P_J|\Psi_J\rangle \to \hat{p}|\Psi\rangle \tag{21}$$

Thus when L is large compared to the Compton wavelength λ_c we can use as a very good approximation the Schrödinger–Heisenberg formalism of quantum mechanics with infinite continuous eigenvalue spectra for q and p and suitable boundary conditions on the wave function so that the particle is not found outside the region of confinement. This is just as demanded by the general philosophy of Bohr's correspondence principle.

(c) The dimension of the matrices Q and P obeying Weyl's relation of the type in (18) can be chosen to be an event integer also, say $2N$, and correspondingly the position and momentum spectra can be, respectively, considered to be given by

$$\{-J\epsilon_J, -(J-1)\epsilon_J, \ldots, -\tfrac{1}{2}\epsilon_J, \tfrac{1}{2}\epsilon_J, \ldots, (J-1)\epsilon_J, J\epsilon_J\}$$

and

$$\{-J\eta_J, -(J-1)\eta_J, \ldots, -\tfrac{1}{2}\eta_J, \tfrac{1}{2}\eta_J, \ldots, (J-1)\eta_J, J\eta_J\}$$

with $J = N - 1/2$ and suitable definitions of ϵ_J and η_J following the normal requirements that there should be symmetry between the two opposite directions. But in such a case we are forced to exclude the zero eigenvalue for momentum implying an a priori unreasonable assumption of a sort of inherent eternal motion for the concerned particle. Hence in any case the dimension of the matrices Q and P must be considered to be an odd integer.

3. FDQM IN THREE-DIMENSIONAL SPACE

Let us now consider that the coordinates of the particle are restricted to be within a spherical region of fixed radius ρ. In this case the system of spherical coordinates (r, θ, φ) defined in the usual manner, with the center of the spherical region as the fixed origin, is the proper choice so that the resulting physical picture of the quantized configuration space of the particle is invariant under rotations of the coordinate frame about the origin. Then the theory of FDQM in three-dimensional space can be developed in close analogy with the one-dimensional case detailed above. The resulting formalism of the FDQM in three-dimensional space consists of three-dimensional versions, obtained by obvious generalization, of the rules (iii) and (iv) of Section 2 above for the case of one-dimensional space and the replacement of the corresponding rules (i) and (ii) by the following:

(i) The eigenvalues of the radial coordinate r of the particle form a discrete and finite set $\{r_{Jn}\}$ given by

$$r_{Jn} = n\epsilon_J, \qquad n = 1, 2, \ldots, 2J+1 \tag{22}$$

$$\epsilon_J = \left(\frac{2\pi}{2J+1}\right)^{1/2} \lambda\!\!\!^{-} \tag{23}$$

and characterized by an integer $J \geqslant 0$, the "space quantum number," such that

$$(2J+1)\epsilon_J \leqslant \rho < (2J+3)\epsilon_{J+1} \tag{24}$$

or

$$J = \text{integer part of } \left(\frac{\rho^2}{4\pi\lambda\!\!\!^{-2}} - \frac{1}{2} \right) \tag{25}$$

Hence the matrix operator for the radial coordinate r, say R_J, is given in position representation by

$$\langle n | R_J | n' \rangle = n\epsilon_J \delta_{nn'}, \qquad n, n' = 1, 2, \ldots, 2J+1 \tag{26}$$

where ϵ_J and J are determined by (23) and (25). The angular coordinates θ and φ take all values in the ranges $[0, \pi]$ and $[0, 2\pi]$, respectively, as usual. Thus in position representation the state vector of the particle is a $(2J+1)$-component vector where each component is a function of θ and φ.

(ii) The radial-momentum operator P_{rJ}, conjugate to R_J, with respect to the corresponding Weyl relation, is given in position representation by

$$\langle n | P_{rJ} | n' \rangle = \begin{cases} 0 & \text{if } n = n' \\ \dfrac{in'\eta_J}{2n} \operatorname{cosec}\left[\dfrac{2\pi J(n-n')}{2J+1} \right] & \text{if } n \neq n' \end{cases}$$

$$n, n' = 1, 2, \ldots, 2J+1 \tag{27}$$

with the spectrum

$$p_{rJn} = n\eta_J, \qquad n = -J, -J+1, \ldots, -1, 0, 1, \ldots, J-1, J \tag{28}$$

where

$$\eta_J = \left(\frac{2\pi}{2J+1} \right)^{1/2} mc \tag{29}$$

Since the angular coordinates θ and φ are not quantized the operators involving only θ, φ and derivatives with respect to them can be retained in the same form as in the current quantum theory to represent the corresponding observables in FDQM. For example, the square of angular momentum L^2 will be represented in FDQM also by the operator

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] \tag{30}$$

It is obvious that the other aspects of the FDQM in one-dimensional space detailed above can be generalized in a straightforward manner to the case of the three-dimensional space. For example the usual nonrelativistic quantum Hamiltonian,

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_r^2 + \frac{\hat{L}^2}{\hat{r}^2}\right) + V(\hat{r}) \tag{31}$$

of the particle moving in a central field of potential $V(r)$ will be now represented by the matrix-differential operator

$$\mathcal{H}_J = \frac{1}{2m}\left(P_{rJ}^2 + R_J^{-2}\hat{L}^2\right) + V(R_J) \tag{32}$$

operating on the $(2J+1)$-component wave function with the elements, $\{\langle n,\theta,\varphi|\Psi_J(t)\rangle | n = 1,2,\ldots,2J+1\}$. All the eigenvalues of $\mathcal{H}_J$ can be shown to be real when $V(R_J)$ is real. In general while forming the operators in FDQM from the corresponding normal quantum mechanical operators in (r,θ,φ) representation only the radial coordinate and momentum operators are to be replaced, according to the rule,

$$\hat{r} \to R_J, \qquad \hat{p}_r \to P_{rJ} \tag{33}$$

since in accordance with the correspondence principle

$$\begin{gathered} \text{as } \rho \to \infty, \qquad J \to \infty, \qquad \epsilon_J \to 0, \qquad \eta_J \to 0 \\ \langle n|R_J|n'\rangle \to \langle r|\hat{r}|r'\rangle = r\delta(r-r') \\ \langle n|P_{rJ}|n'\rangle \to \langle r|\hat{p}_r|r'\rangle = \left[-\frac{i\hbar}{r}\left(\frac{\partial}{\partial r}\right)r\right]\delta(r-r') \end{gathered} \tag{34}$$

and the eigenvalue spectra of the angular coordinates θ and φ are not affected by restricting the value of r to be $\leqslant \rho$. The dimension of the matrices R_J and P_{rJ} must also be odd for the reason similar to that mentioned in the case of one-dimensional space. It can be seen easily as in the one-dimensional case that when $\rho \gg \lambda_c$ the usual Schrödinger–Heisenberg formalism of quantum mechanics becomes a very good approximation to the FDQM in three-dimensional space. In the application of the theory of FDQM to particle physics it is possible that we would have to consider J or ρ not as a fixed quantity but as a stochastic variable with a suitable probability distribution. For the description of a many-particle system in FDQM the direct product formalism must be used as usual in the normal quantum theory.

4. AN EXAMPLE OF THE FDQM: A SPHERICAL HARMONIC OSCILLATOR

In conclusion let us consider a spherical harmonic oscillator as an example of the application of the formalism of FDQM. Let the particle of mass m be restricted to move within a spherical region of radius ρ such that

$$\left(\frac{3}{2\pi}\right)^{1/2} \leq \frac{\rho}{\lambda_c} < \left(\frac{5}{2\pi}\right)^{1/2} \qquad \text{or } J=1 \tag{35}$$

and subjected to a harmonic force field of potential

$$V(r)=\tfrac{1}{2}m\omega^2 r^2 \tag{36}$$

Then according to the rules of FDQM stated above, any stationary state vector of the particle is to be represented in this case by

$$|\Psi\rangle = \begin{pmatrix} \psi_1(\theta,\varphi) \\ \psi_2(\theta,\varphi) \\ \psi_3(\theta,\varphi) \end{pmatrix} \tag{37}$$

and the corresponding time-independent Hamiltonian operator is given by

$$\mathcal{H} = \frac{1}{2m}\left(P_{r1}^2 + R_1^{-2}\hat{L}^2\right) + \frac{1}{2}m\omega^2 R_1^2 \tag{38}$$

where

$$R_1 = \epsilon_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad \epsilon_1 = \left(\frac{2\pi}{3}\right)^{1/2} \lambda \tag{39}$$

$$P_{r1} = \frac{i\eta_1}{\sqrt{3}} \begin{pmatrix} 0 & -2 & 3 \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}, \qquad \eta_1 = \left(\frac{2\pi}{3}\right)^{1/2} mc \tag{40}$$

as obtained from (23), (26), (27) and (29). Following the usual factorization procedure in the central-field problems let us write an eigenvector of $\mathcal{H}$ in (38) as

$$|\Psi\rangle = R_1^{-1}|\chi_l\rangle Y_{lm}(\theta,\varphi) = R_1^{-1} \begin{pmatrix} \chi_{l1} \\ \chi_{l2} \\ \chi_{l3} \end{pmatrix} Y_{lm}(\theta,\varphi) \tag{41}$$

Now the eigenvalue equation for $\mathcal{H}$, namely,

$$\mathcal{H}|\Psi\rangle = E|\Psi\rangle \tag{42}$$

becomes

$$H_l|\chi_l\rangle = E_l|\chi_l\rangle, \qquad l=0,1,2,\ldots \tag{43}$$

with

$$H_l = \begin{pmatrix} 2\alpha+\beta+\gamma & -\alpha & -\alpha \\ -\alpha & 2\alpha+\frac{1}{4}\beta+4\gamma & -\alpha \\ -\alpha & -\alpha & 2\alpha+\frac{1}{9}\beta+9\gamma \end{pmatrix}$$

$$\alpha = \frac{\eta_1^2}{2m}, \qquad \beta = \frac{9l(l+1)\eta_1^2}{8\pi^2 m}, \qquad \gamma = \frac{1}{2}m\omega^2\epsilon_1^2. \tag{44}$$

Then the three possible eigenvalues for E_l can be shown to be given by

$$E_{l1} = \alpha\left[2+14x+49y-2z\cos\left(\frac{\pi-\xi}{3}\right)\right] \tag{45}$$

$$E_{l2} = \alpha\left[2+14x+49y-2z\cos\left(\frac{\pi+\xi}{3}\right)\right] \tag{46}$$

$$E_{l3} = \alpha\left[2+14x+49y-2z\cos\left(\frac{\xi}{3}\right)\right] \tag{47}$$

where

$$x = \left(\frac{\hbar\omega}{mc^2}\right)^2, \qquad y = \frac{l(l+1)}{16\pi^2}$$

$$z = \left(1+49x^2+889y^2-362xy\right)^{1/2}$$

$$\xi = \text{minimum value of } \cos^{-1}\left[\frac{143x^3+24013y^3+399x^2y-10731xy^2-1}{z^3}\right] \tag{48}$$

Here the necessary condition that $|\cos\xi| \leqslant 1$ is satisfied as a consequence of the fact that all the eigenvalues of the real symmetric matrix H_l must be real.

When ω and l are small or $x \approx y \approx 0$, such that in the right-hand sides of (48) the terms containing x and y can be neglected in comparison with 1, we have $z \approx 1$, $\xi \approx \pi$ and hence

$$E_{l1} \approx \frac{49mc^2 l(l+1)}{144\pi} + \frac{14\pi\hbar^2\omega^2}{9mc^2}$$

$$E_{l2} \approx E_{l3} \approx \frac{\pi}{3} mc^2 + E_{l1} \tag{49}$$

These results demonstrate the remarkable difference that can arise between the applications of the usual quantum theory and the FDQM when the latter is actually more appropriate, i.e., ρ/λ_c is not very large. But when ρ/λ_c is very large the results of FDQM should not of course differ very much from those of the usual quantum theory.

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